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Journal of Geometry and Physics 43 (2002) 371–383

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Observer-dependent Gauss–Codazzi formalism for null hypersurfaces in the space–time

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Received 9 August 2001; received in revised form 9 January 2002

Abstract

We introduce a new Gauss–Codazzi framework for null hypersurfaces in the space–time. First, with the use of space–time splitting techniques, and working within the framework of general coordinates of the ambient space–time, we generalize the second fundamental form and the Ricci and Gauss–Codazzi formulae of a non-null hypersurface Σ to a neighbourhood of it. Then in a similar way we introduce a second fundamental form analogue for the null hypersurface case, and deduce the corresponding Ricci and Gauss–Codazzi formulae.

MSC: 53B20; 53B50

PACS: 02.40.Hw; 04.20.-q

Subj. Class.: Differential geometry; General relativity

Keywords: Space–time splitting; Null hypersurfaces; Extrinsic curvature

1. Introduction

The traditional Gauss–Codazzi formalism belongs to the framework of the parametric equations of a non-null hypersurface Σ. It describes the differential geometry induced by the ambient space–time, in terms of the second fundamental form (extrinsic curvature) of Σ, by means of the inner Ricci formula and the Gauss–Codazzi identities. Clearly, such an approach cannot be directly applied to null hypersurfaces, since the induced metric is in this case degenerate, and the usual induced Levi-Civita connection is undefined.

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PII: S0393-0440(02)00025-6

In this paper, we construct an extrinsic curvature analogue for null hypersurfaces, and deduce the corresponding Ricci formula and Gauss–Codazzi identities. The framework is that of generic ambient coordinates, of the Cartesian equation of the hypersurface and of space–time splitting.

First, as a preliminary problem, we extend the usual second fundamental form of a non-null hypersurface to a neighbourhood of Σ by a method of projection and study the corresponding generalizations of the Ricci and Gauss–Codazzi formulae.

Then we consider a similar procedure for a null hypersurface, with the help of the "transverse" double projection relative to a reference frame; this leads to an observer-dependent definition of second fundamental form analogue, and to the corresponding Ricci and Gauss–Codazzi formulae, which do not have a traditional counterpart.

Examples of mathematical theories of characteristic hypersurfaces which have a Gauss–Codazzi analogue can be found in the literature. Replacements for the Levi-Civita connection on Σ can be defined directly (see, e.g. [6,14–17,30,32]), or be induced by the so-called screen-distribution, i.e. an arbitrary completion of the parabolic-degenerate tangent space of Σ (see, e.g. [2,8] and the more concise [11]).

Be they "natural" or not, such extensions involve arbitrary auxiliary geometric structures; so it is clear that there is no a unique way for extending the Gauss–Codazzi formalism to the null case. So, why to bother for introducing a new one? We are mainly motivated by two reasons.

First, in the cited literature attention is focused on the parametric equations of the hypersurface, and inner coordinates and holonomic 3-basis are used. This (perfectly legitimate) point of view is not always the most convenient; some applications in fact may be more naturally cast in the framework of ambient space–time, i.e. with the use of the Cartesian equation of the hypersurface and of general ambient coordinates. In this case "translation" in terms of extrinsic curvature and 3-parameters may not be a trivial nor a natural thing to do; this is the reason for constructing a theory which completely lies in the framework of ambient space–time.

Second, we mean to introduce the observer's point of view, making use of the space–time splitting with respect to a generic reference frame, to show that non-uniqueness in the null hypersurface extension of the Gauss–Codazzi framework in principle can be interpreted in terms of different physical measures relative to different observers. This interpretation can be very useful for subsequent application to physical evolution problems of general relativity involving null hypersurfaces, as it will be showed elsewhere.

In fact, the space–time splitting method allows one to disentangle the Gauss–Codazzi formalism from the 3-parameters framework, focusing on the Cartesian equation of the hypersurface and using general coordinates of the ambient space–time. With this method it is possible to replace the traditional second fundamental form of a non-characteristic hypersurface with a more general field, which has support on a whole neighbourhood of the hypersurface, and is defined (uniquely) in a geometrical way. Besides the generalization, we then are led in a natural way to the definition of a complete Gauss–Codazzi formalism also for the characteristic case. In this case the same operational role which is elsewhere played, for example, by a screen-distribution, is here played by a reference frame (i.e. an auxiliary congruence of time-like lines), thus automatically giving some physical meaning, in terms of the observer's acceleration, deformation and vorticity, to the frame-dependent geometrical objects which are introduced. Here non-uniqueness is the reflection of different measurements made by different observers, but we have formal invariance with respect to the choice of the reference frame. The generic reference frame can then be specialized in order for additional properties to hold.

The present approach is rather simple, and completely lies in the framework of local differential geometry and tensor algebra (see [22] for a Newmann–Penrose spin-coefficients formulation of the null hypersurface Gauss–Codazzi analogue).

The geometry of three-dimensional hypersurfaces in space–time is involved in a number of fundamental physical problems in general relativity.

For example space-like hypersurfaces (i.e. with a time-like normal vector) are the non-characteristic manifolds of the Cauchy problem for the Einstein gravitational equations; here the standard method for encoding initial data requires the extrinsic curvature formalism (see, e.g. [7, pp. 143 and 152; 23, pp. 509–518]).

Time-like and light-like hypersurfaces (i.e. with a space-like and a light-like normal vector, respectively), on the other hand, play the fundamental role of space–time interfaces. In general relativity in fact an important problem is that of the continuous (but not C^1) match of two solutions of the Einstein equations across a discontinuity hypersurface; classic applications are for example: (1) the study of the propagation of gravitational shock waves [3,5,9,18,19], (2) the study of the evolution of a thin shell of matter [1,4,12,20,27–29], (3) the modelling of gravitational collapse [26,31]. The most used theory of discontinuity hypersurfaces was formulated in its actual form by Israel [12], Mansouri and Khorrami [21] and Papapetrou and Hamoui [27], and is again based on the parametric equations of the hypersurface and on the Gauss–Codazzi formalism. Clearly, such theory finds its natural application in case the hypersurface is time-like (see, e.g. [24,29,31]). A method ad hoc for handling the characteristic (i.e. light-like) situation was introduced by Barrabes and Israel [1] (see also [25]), but the traditional Gauss–Codazzi formulae are still missing with this method. Nevertheless, it is possible to apply our formalism to this problem, thus recovering the traditional features also in the null case. For the sake of brevity, however, here we limit to regular metrics; application of our extension of the Gauss–Codazzi formalism to the problem of gravitational discontinuity hypersurfaces, and a detailed study of Barrabes and Israel theory, will be discussed elsewhere.

This paper is organized as follows. In Section 2, definitions are stated and the concept of reference frame is briefly introduced. In Section 3, the case of non-null hypersurfaces is studied. The orthogonal splitting of the ambient space–time, with respect to the unit normal vector field, is considered. The generalized second fundamental form of a time-like hypersurface is introduced; for a space-like hypersurface this is replaced by the deformation tensor of the normal reference frame. In both cases the corresponding generalized Ricci and Gauss–Codazzi formulae are proved. In Section 4 the case of a null hypersurface is studied. The transverse-splitting, relative to a reference frame, is considered. The transverse second fundamental form analogue, which is observer-dependent, but formally invariant with respect to the choice of the reference frame, is defined; the corresponding Ricci and Gauss–Codazzi formulae are deduced. Relation with the method of screen-distributions is then briefly considered.

2. Reference frames

Let V_4 be an oriented differentiable manifold of dimension 4, class C^3, provided with a strictly hyperbolic metric of signature $-+++$ and class C^2. Let $\eta_{\alpha\beta\gamma\delta}$ denote the unit volume 4-form and ∇ the covariant derivative. Let $\Omega \subset V_4$ be an open connected subset with compact closure. Let units be chosen in order to have the speed of light in empty space $c \equiv 1$. Greek indices run from 0 to 3; Latin indices run from 1 to 3.

The Riemann curvature tensor R is defined by the Ricci formula:

$$(\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta) V^\sigma = R_{\alpha\beta\rho}{}^\sigma V^\rho. \tag{1}$$

Let $\Sigma \subset \Omega$ be a regular hypersurface of equation $f(x) = 0$. Let $f \in C^3(\Omega)$. Finally, let $\ell_\alpha \equiv \partial_\alpha f$ denote the gradient of f.

Let u^α be a time-like unit vector field of class $C^2(\Omega)$; we call such a field a reference frame, or an observer. The corresponding three-dimensional spatial metric is $h(u)_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$. The fields u and $h(u)$ define locally a $1+3$ splitting of the space–time in Ω. In the following the suffix "(u)" will be consistently used to denote spatial fields with respect to u, i.e. fields normal to u; such fields are also said to belong to the so-called local rest space of u [13].

The covariant derivative of u can be uniquely decomposed in the following way [10,13]:

$$\nabla_\alpha u_\beta = \eta(u)_{\alpha\beta}{}^\mu \omega(u)_\mu + \Theta(u)_{\alpha\beta} - u_\alpha a(u)_\beta, \tag{2}$$

where $\eta(u)$ is the spatial unit volume 3-form, $\omega(u)$ the vorticity vector, $\Theta(u)$ the expansion (symmetric) tensor and $a(u)$ the acceleration vector. We have

$$\eta(u)_{\alpha\beta\rho} = \eta_{\sigma\alpha\beta\rho} u^\sigma, \tag{3}$$

$$\omega(u)^\alpha = \tfrac{1}{2} \eta(u)^{\alpha\mu\nu} \nabla_\mu u_\nu, \tag{4}$$

$$\Theta(u)_{\alpha\beta} = h(u)_{(\alpha}{}^\nu h(u)_{\beta)}{}^\mu \nabla_\nu u_\mu, \tag{5}$$

$$a(u)^\alpha = u^\nu \nabla_\nu u^\alpha. \tag{6}$$

A reference frame u is said to be normal if we have $\omega(u) = 0$. In this case the corresponding time-like congruence is integrable.

3. Non-null hypersurfaces

In the neighbourhood of a non-null hypersurface it is easy to split the space–time into the sum of the normal space (i.e. collinear with the normal vector, be it space-like or time-like) and the "tangent" space (i.e. orthogonal to the normal vector). The interesting point is that the tangent projection of the covariant gradient of the unit normal vector field is a generalization of the extrinsic curvature, as we are going to see.

3.1. Time-like hypersurfaces

Suppose $(\ell \cdot \ell) > 0$ (Σ time-like). We define the space-like unit normal vector $N_\alpha = (\ell \cdot \ell)^{-(1/2)} \ell_\alpha$ and the corresponding three-dimensional tangent metric $h(N)_{\alpha\beta} = g_{\alpha\beta} - N_\alpha N_\beta$. Such vector field and metric field actually define a 1 + 3 splitting of the space–time in the whole Ω, where in particular $h(N)$ is tangent to the family of hypersurfaces of equation $f = \text{const.}$ and N the unit tangent of the corresponding integrable congruence of space-like curves. In the following the suffix "(N)" will be consistently used to denote tangent fields, i.e. fields normal to N.

Let us define the inner covariant derivative operator $\nabla(N)$ of a generic tangent tensor by projection on Σ; for example if $V(N)$ is a generic vector and $W(N)$ a generic 2-tensor tangent to Σ, we have

$$\nabla(N)_\alpha V(N)_\beta = h(N)_\alpha{}^\nu h(N)_\beta{}^\mu \nabla_\nu V(N)_\mu, \tag{7}$$

$$\nabla(N)_\alpha W(N)_{\beta\rho} = h(N)_\alpha{}^\nu h(N)_\beta{}^\mu h(N)_\rho{}^\sigma \nabla_\nu W(N)_{\mu\sigma}, \tag{8}$$

and similarly for tangent tensors of higher rank. Then consider the following tensor field:

$$K(N)_{\alpha\beta} = +h(N)_\alpha{}^\nu h(N)_\beta{}^\mu \nabla_\nu N_\mu = -h(N)_\alpha{}^\nu h(N)_\beta{}^\mu N^\rho \nabla_\nu h(N)_{\mu\rho}, \tag{9}$$

$K(N)_{\alpha\beta}$ is symmetric and regularly discontinuous on Σ. We have

$$\nabla_\alpha N_\beta = K(N)_{\alpha\beta} + N_\alpha \nabla_N N_\beta, \tag{10}$$

where $\nabla_N N_\beta = N^\nu \nabla_\nu N_\beta$.

On each side of Σ, $K(N)$ is a regular extension of the second fundamental form of the hypersurface. To see it, suppose Σ is described by three parameters ξ^i $(i = 1, 2, 3)$ such that the parametric equations of the hypersurface are in the following form:

$$x^\alpha = x^\alpha(\xi^1, \xi^2, \xi^3), \tag{11}$$

where x^α are C^1 functions of ξ^i. If e_i is the corresponding 3-basis on Σ, i.e. $(e_i)^\alpha = \partial x^\alpha / \partial \xi^i$, then the usual second fundamental form of Σ is defined by

$$K_{ij} = e_i \frac{\partial N}{\partial \xi^j}, \tag{12}$$

and one finds out

$$K_{ij} = \frac{\partial x^\nu}{\partial \xi^i} \frac{\partial x^\mu}{\partial \xi^j} \nabla_\nu N_\mu. \tag{13}$$

One can consider an extension outside Σ of the set ξ^i of coordinates (see, e.g. [27]), by letting, for example $x^i = \xi^i$ and $x^0 = f$, where $f = 0$ is the equation of Σ. With such regular coordinates one has $N_\alpha = (\ell \cdot \ell)^{-1/2} \delta_\alpha{}^0$ and $(e_i)^\alpha = \delta^\alpha{}_i = h(N)^\alpha{}_i$. In such a chart therefore $\nabla(N)_j$ corresponds to the ordinary inner covariant derivative along the directions of ξ^j, i.e. the projection of the ambient Levi-Civita connection coincides on Σ with the induced connection. Similarly, from (9) we see that the 3-parameters component of $K(N)$ is equal on Σ to the usual second fundamental form. But $K(N)$ is defined by projection, in a general and geometrical way which is invariant with respect to the ambient coordinates (and with respect to the parameters). For this reason in the following, we will actually replace the traditional second fundamental form of Σ with the extended field $K(N)$ defined above, and we will use general coordinates.

It is fundamental to note that the *Gauss–Codazzi* framework can be generalized to a neighbourhood of Σ and to the extended field (9). We have in fact the following theorem.

Theorem 1. *Consider a vector field $V(N)$ tangent to Σ (i.e. $(V \cdot N) = 0$); we have the following tangent Ricci formula*:

$$2\nabla(N)_{[\beta}\nabla(N)_{\alpha]}V(N)^{\sigma} = (R_{\Sigma})_{\alpha\beta\rho}{}^{\sigma}V(N)^{\rho} \tag{14}$$

with R_Σ given by the following first Gauss–Codazzi identity:

$$(R_{\Sigma})_{\alpha\beta\rho}{}^{\sigma} = R(N)_{\alpha\beta\rho}{}^{\sigma} + K(N)_{\beta}{}^{\sigma}K(N)_{\alpha\rho} - K(N)_{\alpha}{}^{\sigma}K(N)_{\beta\rho}, \tag{15}$$

where $R(N)$ is the tangent projection of R: $R(N)_{\alpha\beta\rho}{}^{\sigma} = h(N)_{\alpha}{}^{\nu}h(N)_{\beta}{}^{\mu}h(N)_{\rho}{}^{\lambda}h(N)^{\gamma\sigma}$ $R_{\nu\mu\lambda\gamma}$. Moreover, we have the following second Gauss–Codazzi identity:

$$h(N)_{\alpha}{}^{\nu}h(N)_{\beta}{}^{\mu}h(N)_{\sigma}{}^{\gamma}N^{\rho}R_{\nu\mu\rho\gamma} = 2\nabla(N)_{[\beta}K(N)_{\alpha]\sigma}. \tag{16}$$

Proof. Consider the following identity:

$$\nabla_{\beta}\nabla_{\alpha}V(N)^{\sigma} = \nabla_{\beta}\{(h(N)_{\alpha}{}^{\nu} + N_{\alpha}N^{\nu})(h(N)^{\sigma\mu} + N^{\sigma}N^{\mu})\nabla_{\nu}V(N)_{\mu}\}. \tag{17}$$

The completely tangent component (i.e. all indices projected by means of the tangent metric $h(N)$) of the left-hand side is $h(N)_{\beta}{}^{\nu}h(N)_{\alpha}{}^{\mu}h(N)^{\sigma\mu}\nabla_{\nu}\nabla_{\mu}V(N)_{\lambda}$, while that of the second-hand side turns out to be

$$\nabla(N)_{\beta}\nabla(N)_{\alpha}V(N)^{\sigma} + K(N)_{\beta\alpha}N^{\nu}h(N)^{\sigma\mu}\nabla_{\nu}V_{\mu} - K(N)_{\beta}{}^{\sigma}K(N)_{\alpha\mu}V(N)^{\mu}. \tag{18}$$

Then, by anti-symmetrization, and from the ordinary space–time Ricci formula (1), we have (15), as wished. As for (16), again from (1) we have

$$\begin{aligned}\tfrac{1}{2}R_{\alpha\beta\rho\sigma}N^{\rho} &= \nabla_{[\beta}\nabla_{\alpha]}N_{\sigma} = \nabla_{[\beta}(\nabla(N)_{\alpha]}N_{\sigma} + N_{\alpha]}N^{\nu}\nabla_{\nu}N_{\sigma})\\ &= \nabla_{[\beta}K(N)_{\alpha]\sigma} + \nabla_{[\beta}N_{\alpha]}N^{\nu}\nabla_{\nu}N_{\sigma}\\ &\quad + N_{[\alpha}\nabla_{\beta]}N^{\nu}\nabla_{\nu}N_{\sigma} + N_{[\alpha}N^{\nu}\nabla_{\beta]}\nabla_{\nu}N_{\sigma},\end{aligned} \tag{19}$$

the completely tangent component of the last term is $\nabla(N)_{[\beta}K(N)_{\alpha]\sigma}$, which leads to (16). Thus our proof is completed. □

3.2. Space-like hypersurfaces

In the space-like situation the role of extrinsic curvature is actually played by the deformation tensor of the normal reference frame, as we are going to see. This case was studied along similar lines in [7, pp. 143 and 252] (see also [23, pp. 517–518]), with a different notation.

Suppose $(\ell \cdot \ell) < 0$ (Σ space-like). We define the time-like unit normal vector $U_\alpha = [-(\ell \cdot \ell)]^{-(1/2)}\ell_\alpha$; the corresponding three-dimensional spatial tangent metric is $h(U)_{\alpha\beta} =$

$g_{\alpha\beta} + U_\alpha U_\beta$. We call U the normal reference frame of the hypersurface. Such vector field and metric field actually define a $1+3$ splitting of the space–time in the whole Ω, where in particular, $h(U)$ is tangent to the family of hypersurfaces of equation $f = \text{const.}$ and U is the unit tangent of the corresponding integrable congruence of time-like curves.

Let us define the inner covariant derivative operator $\nabla(U)$ of a generic tangent tensor by projection on Σ; for example if $V(U)$ is a generic vector and $W(U)$ a generic 2-tensor tangent to Σ we have

$$\nabla(U)_\alpha V(U)_\beta = h(U)_\alpha{}^\nu h(U)_\beta{}^\mu \nabla_\nu V(U)_\mu, \tag{20}$$

$$\nabla(U)_\alpha W(U)_{\beta\rho} = h(U)_\alpha{}^\nu h(U)_\beta{}^\mu h(U)_\rho{}^\sigma \nabla_\nu W(U)_{\mu\sigma}, \tag{21}$$

and similarly for tangent tensors of higher rank.

Since U is a normal reference frame from (2) we have the following decomposition:

$$\nabla_\alpha U_\beta = \Theta(U)_{\alpha\beta} - U_\alpha a(U)_\beta. \tag{22}$$

We have the following theorem.

Theorem 2. *Consider a vector field $V(U)$ tangent to Σ (i.e. $(V \cdot U) = 0$); we have the following tangent Ricci formula*:

$$2\nabla(U)_{[\beta}\nabla(U)_{\alpha]} V(U)^\sigma = (R_\Sigma)_{\alpha\beta\rho}{}^\sigma V(U)^\rho \tag{23}$$

with R_Σ given by the following first Gauss–Codazzi identity:

$$(R_\Sigma)_{\alpha\beta\rho}{}^\sigma = R(U)_{\alpha\beta\rho}{}^\sigma - \Theta(U)_\beta{}^\sigma \Theta(U)_{\alpha\rho} + \Theta(U)_\alpha{}^\sigma \Theta(U)_{\beta\rho}, \tag{24}$$

where $R(U)$ is the spatial projection of R: $R(U)_{\alpha\beta\rho}{}^\sigma = h(U)_\alpha{}^\nu h(U)_\beta{}^\mu h(U)_\rho{}^\lambda h(U)^{\gamma\sigma}$ $R_{\nu\mu\lambda\gamma}$. Moreover, we have the following second Gauss–Codazzi identity:

$$h(U)_\alpha{}^\nu h(U)_\beta{}^\mu h(U)_\sigma{}^\gamma U^\rho R_{\nu\mu\rho\gamma} = 2\nabla(U)_{[\beta}\Theta(U)_{\alpha]\sigma}. \tag{25}$$

Proof. Consider the following identity:

$$\nabla_\beta \nabla_\alpha V(U)^\sigma = \nabla_\beta\{(h(U)_\alpha{}^\nu - U_\alpha U^\nu)(h(U)^{\sigma\mu} - U^\sigma U^\mu)\nabla_\nu V(U)_\mu\}. \tag{26}$$

The completely tangent component (i.e. all indices projected by means of the tangent metric $h(U)$) of the left-hand side is: $h(U)_\beta{}^\nu h(U)_\alpha{}^\mu h(U)^{\sigma\mu}\nabla_\nu\nabla_\mu V(U)_\lambda$, while that of the second-hand side turns out to be

$$\nabla(U)_\beta\nabla(U)_\alpha V(U)^\sigma - \Theta(U)_{\beta\alpha} N^\nu h(U)^{\sigma\mu}\nabla_\nu V_\mu + \Theta(U)_\beta{}^\sigma \Theta(U)_{\alpha\mu} V(N)^\mu. \tag{27}$$

Then, by anti-symmetrization, and from the ordinary space–time Ricci formula (1), we have (24), as wished. As for (25), again from (1) we have

$$\begin{aligned}\tfrac{1}{2} R_{\alpha\beta\rho\sigma} U^\rho &= \nabla_{[\beta}\nabla_{\alpha]} U_\sigma = \nabla_{[\beta}(\Theta_{\alpha]\sigma} - U_{\alpha]} a(U)_\sigma) \\ &= \nabla_{[\beta}\Theta(U)_{\alpha]\sigma} - \nabla_{[\beta} U_{\alpha]} a(U)_\sigma + U_{[\alpha}\nabla_{\beta]} a(U)_\sigma,\end{aligned} \tag{28}$$

the completely tangent component of the last term is $\nabla(U)_{[\beta}\Theta(U)_{\alpha]\sigma}$, which leads to (25). Thus our proof is completed. □

Note that for R_Σ, with a slight abuse of notation, we have used the same symbol of the time-like case.

4. Null hypersurfaces

In the neighbourhood of a null hypersurface the space of normal-orthogonal vectors is not a complement of the tangent space. Nevertheless, the problem of splitting the space–time can be solved by introducing an arbitrary time-like congruence (reference frame), which permits us, as we are going to see, to define a relative two-dimensional tangent space, which we name transverse for the sake of brevity.

Suppose $(\ell \cdot \ell) = 0$. Let u^α be a reference frame; we can write the following decomposition:

$$\ell^\alpha = -(u \cdot \ell) L^\alpha, \quad L^\alpha = u^\alpha + n(u)^\alpha. \tag{29}$$

In particular, we do not require the gradient vector to be normalized, i.e. $(u \cdot \ell)$ to be constant. The null vector L is determined by the couple ℓ, u, so a more precise notation should be $L_{\ell,u}$, however, for the sake of brevity we avoid such notation in the following. For the spatial vector $n(u)$, we have $n(u)^\alpha = h(u)^{\alpha\beta} L_\beta$. In the following, however, again for the sake of brevity, we will drop the suffix (u) and simply denote this vector by n.

Now let us introduce the transverse metric $h(u,n)_{\alpha\beta}$, orthogonal to both u and n (and therefore also to ℓ), defined as follows:

$$h(u,n)_{\alpha\beta} = h(u)_{\alpha\beta} - n_\alpha n_\beta = g_{\alpha\beta} + u_\alpha u_\beta - n_\alpha n_\beta = g_{\alpha\beta} + L_\alpha u_\beta - n_\alpha L_\beta. \tag{30}$$

We thus have in Ω a "$1+1+2$" splitting of the space–time, generated by u, n, and $h(u,n)$. In the following, the suffix "(u,n)" will be consistently used to denote transverse fields, i.e. fields which are normal to both u and n.

Consider for a moment a couple of transverse unit vectors $a(u,n)$, $b(u,n)$ forming a spatial orthonormal basis together with $n(u)$; any transverse tensor field has non-vanishing components along the various possible tensor products of a and b only. For example, we have

$$h(u,\nu)_{\alpha\beta} = a(u,\nu)_\alpha a(u,\nu)_\beta + b(u,\nu)_\alpha b(u,\nu)_\beta, \tag{31}$$

$$\eta(u,\nu)_{\alpha\beta} = a(u,\nu)_\alpha b(u,\nu)_\beta - b(u,\nu)_\alpha a(u,\nu)_\beta. \tag{32}$$

The further splitting of $\eta(u)$ along n and the transverse 2-space relative to u and n is the following:

$$\eta(u)_{\alpha\beta\rho} = 2n_{[\alpha}\eta(u,n)_{\beta]\rho} + n_\rho \eta(u,n)_{\alpha\beta}, \tag{33}$$

where the transverse component $\eta(u,n)_{\alpha\beta} = \eta(u)_{\sigma\alpha\beta} n^\sigma$ is such that

$$\eta(u,n)_{\lambda\mu}\eta(u,n)_{\sigma\nu} = 2h(u,n)_{\sigma[\lambda} h(u,n)_{\mu]\nu}. \tag{34}$$

The splitting of $\omega(u)$, $\Theta(u)$ and $a(u)$ in turn gives rise in a natural way to the following transverse fields:

$$\Omega(u,n)_\alpha = \eta(u,n)_{\alpha\sigma}\omega(u)^\sigma, \tag{35}$$

$$\Theta(u,n)_{\alpha\beta} = h(u,n)_{\alpha}{}^{\nu} h(u,n)_{\beta}{}^{\mu} \Theta(u)_{\nu\mu}, \tag{36}$$

$$K_u(u,n)_{\alpha\beta} = h(u,n)_{\alpha}{}^{\nu} h(u,n)_{\beta}{}^{\mu} \nabla_{\nu} u_{\mu} = \Theta(u,n)_{\alpha\beta} + (\omega(u)\cdot n)\eta(u,n)_{\alpha\beta}, \tag{37}$$

$$\Theta(u,n)_{\alpha} = n^{\nu} h(u,n)_{\alpha}{}^{\mu} \Theta(u)_{\nu\mu}, \tag{38}$$

$$\Theta(u,n) = n^{\nu} n^{\mu} \Theta(u)_{\nu\mu}, \tag{39}$$

$$a(u,n)_{\alpha} = h(u,n)_{\alpha}{}^{\nu} a(u)_{\nu}, \tag{40}$$

which are such that

$$\begin{aligned}\nabla_{\alpha} u_{\beta} = {} & K_u(u,n)_{\alpha\beta} + (\Theta(u,n)_{\alpha} - \Omega(u,n)_{\alpha}) n_{\beta} + (\Theta(u,n)_{\beta} + \Omega(u,n)_{\beta}) n_{\alpha} \\ & - a(u,n)_{\beta} u_{\alpha} - (a(u)\cdot n) u_{\alpha} n_{\beta} + \Theta(u,n) n_{\alpha} n_{\beta}.\end{aligned} \tag{41}$$

The splitting of $\nabla_{\alpha} n_{\beta}$, as a consequence of the symmetry of $\nabla_{\alpha}\ell_{\beta}$, from (29) turns out to give rise to two further transverse fields only, i.e.

$$K_n(u,n)_{\alpha\beta} = h(u,n)_{\alpha}{}^{\nu} h(u,n)_{\beta}{}^{\mu} \nabla_{\nu} n_{\mu}, \tag{42}$$

$$\nabla_u(u,n) n_{\alpha} = h(u,n)_{\alpha}{}^{\nu} u^{\mu} \nabla_{\mu} n_{\nu}, \tag{43}$$

which are such that

$$\begin{aligned}\nabla_{\alpha} n_{\beta} = {} & K_n(u,n)_{\alpha\beta} + (\Theta(u,n)_{\alpha} - \Omega(u,n)_{\alpha}) u_{\beta} - \nabla_u(u,n) n_{\beta} u_{\alpha} \\ & - (a(u)\cdot n) u_{\alpha} u_{\beta} + \Theta(u,n) n_{\alpha} u_{\beta} - (\nabla_u(u,n) n_{\beta} + \Theta(u,n)_{\beta} \\ & + \Omega(u,n)_{\beta} + a(u,n)_{\beta}) n_{\alpha}.\end{aligned} \tag{44}$$

Note that the two fields $K_u(u,n)$ and $K_n(u,n)$ are somewhat candidates to play the role of second fundamental form. In fact, if we introduce the transverse covariant derivative $\nabla(u,n)$ in a way similar to (7) and (8), but with $h(u,n)$ in place of $h(N)$, similar to (9) we can write

$$K_u(u,n)_{\alpha\beta} = \nabla(u,n)_{\alpha} u_{\beta}, \tag{45}$$

$$K_n(u,n)_{\alpha\beta} = \nabla(u,n)_{\alpha} n_{\beta}. \tag{46}$$

These fields are in the general case not symmetric, even if by (29) one immediately has that their sum

$$K(u,n)_{\alpha\beta} = K_u(u,n)_{\alpha\beta} + K_n(u,n)_{\alpha\beta} = \nabla(u,n)_{\alpha} L_{\beta} \tag{47}$$

is symmetric, as it is proportional to the completely transverse component of $\nabla_{\alpha}\ell_{\beta}$. Moreover, from (37) and from the symmetry of $K(u,n)$ one finds that $K_u(u,n)$ and $K_n(u,n)$ are both simultaneously symmetric in the particular case the reference frame u is such that

$$n\cdot\omega(u) = \ell\cdot\omega(u) = 0. \tag{48}$$

In this case from (37) we moreover, have $K_u(u,n)_{\alpha\beta} = \Theta(u,n)_{\alpha\beta}$. If condition (48) holds then the transverse curvature tensor satisfies a complete extension of the Ricci formula to the transverse 2-space. In the general case, we have in fact the following theorem.

Theorem 3. *Let $V(u,n)$ be a transverse vector (i.e. $V(u,n)\cdot u = V(u,n)\cdot n = 0$). Then we have the transverse Ricci formula*

$$2\nabla(u,n)_{[\beta}\nabla(u,n)_{\alpha]}V(u,n)^{\sigma} = (R_{\Sigma})_{\alpha\beta\rho}{}^{\sigma}V(u,n)^{\rho} + 2(\omega(u)\cdot n)\eta(u,n)_{\alpha\beta}\nabla_{L}(u,n)V(u,n)^{\sigma}, \tag{49}$$

where $\nabla_L(u,n)V(u,n)^{\sigma} = L^{\nu}h(u,n)^{\mu\sigma}\nabla_{\nu}V(u,n)_{\mu}$ and R_{Σ} is defined by the following first Gauss–Codazzi identity:

$$(R_{\Sigma})_{\alpha\beta\rho\sigma} = R(u,n)_{\alpha\beta\rho\sigma} + K_n(u,n)_{\beta\sigma}K_n(u,n)_{\alpha\rho} - K_n(u,n)_{\alpha\sigma}K_n(u,n)_{\beta\rho} - K_u(u,n)_{\beta\sigma}K_u(u,n)_{\alpha\rho} + K_u(u,n)_{\alpha\sigma}K_u(u,n)_{\beta\rho}, \tag{50}$$

where $R(u,n)_{\alpha\beta\rho\sigma} = h(u,n)_{\alpha}{}^{\nu}h(u,n)_{\beta}{}^{\mu}h(u,n)_{\rho}{}^{\lambda}h(u,n)_{\sigma}{}^{\gamma}R_{\nu\mu\lambda\gamma}$, and where moreover the following pair of second Gauss–Codazzi identities holds:

$$h(u,n)_{\alpha}{}^{\nu}h(u,n)_{\beta}{}^{\mu}h(u,n)_{\sigma}{}^{\gamma}n^{\rho}R_{\nu\mu\rho\gamma} = 2\nabla(u,n)_{[\beta}K_n(u,n)_{\alpha]\sigma} + 2(\Theta(u,n)_{[\alpha} - \Omega(u,n)_{[\alpha})K_u(u,n)_{\beta]\sigma} + 2K_u(u,n)_{[\alpha\beta]}\nabla_u(u,n)n_{\sigma} + 2K_n(u,n)_{[\alpha\beta]}(\Theta(u,n)_{\sigma} + \Omega(u,n)_{\sigma} + a(u,n)_{\sigma}), \tag{51}$$

and

$$h(u,n)_{\alpha}{}^{\nu}h(u,n)_{\beta}{}^{\mu}h(u,n)_{\sigma}{}^{\gamma}u^{\rho}R_{\nu\mu\rho\gamma} = 2\nabla(u,n)_{[\beta}K_u(u,n)_{\alpha]\sigma} + 2(\Theta(u,n)_{[\alpha} - \Omega(u,n)_{[\alpha})K_n(u,n)_{\beta]\sigma} + 2K_u(u,n)_{[\alpha\beta]}a(u,n)_{\sigma} - 2K_n(u,n)_{[\alpha\beta]}(\Theta(u,n)_{\sigma} + \Omega(u,n)_{\sigma}). \tag{52}$$

Proof. Consider the following identity:

$$\nabla_{\beta}\nabla_{\alpha}V(u,n)^{\sigma} = \nabla_{\beta}\{(h(u,n)_{\alpha}{}^{\nu} - u_{\alpha}u^{\nu} + n_{\alpha}n^{\nu})(h(u,n)^{\sigma\mu} - u^{\sigma}u^{\mu} + n^{\sigma}n^{\mu})\nabla_{\nu}V(u,n)_{\mu}\}. \tag{53}$$

The completely transverse component of the left-hand side is given by: $h(u,n)_{\beta}{}^{\nu}h(u,n)_{\alpha}{}^{\mu}h(u,n)^{\sigma\lambda}\nabla_{\nu}\nabla_{\mu}V(u,n)_{\lambda}$, while that of the right-hand side turns out to be

$$\nabla(u,n)_{\beta}\nabla(u,n)_{\alpha}V(u,n)^{\sigma} + (K_u(u,n)_{\beta}{}^{\sigma}K_u(u,n)_{\alpha\mu} - K_n(u,n)_{\beta}{}^{\sigma}K_n(u,n)_{\alpha\mu})V(u,n)^{\mu} + K_n(u,n)_{\beta\alpha}n^{\nu}h(u,n)^{\sigma\mu}\nabla_{\nu}V(u,n)_{\mu} - K_u(u,n)_{\beta\alpha}u^{\nu}h(u,n)^{\sigma\mu}\nabla_{\nu}V(u,n)_{\mu}. \tag{54}$$

Thus, by anti-symmetrization we have (49). As for (57) and (58), they follow rather simply from the Ricci formula (1) and from (41)–(44). □

In particular, we see that R_{Σ} (for which, with a slight abuse of notation, we again used the same symbol of the time-like and space-like case) behaves as a curvature tensor if (48) holds, at least on Σ. This is a scalar condition on the reference frame u, which is still completely arbitrary and operational. Such a condition is satisfied, for example, in the particular case the time-like congruence associated to u is integrable (i.e. $\omega(u) = 0$), but is more general. We say that the reference frame is "normal" with respect to Σ if (48) holds on Σ.

Corollary 4. *In the hypothesis of Theorem 3, if the reference frame u is normal, i.e. (48) holds, then the transverse Ricci formula reduces to*

$$2\nabla(u,n)_{[\beta}\nabla(u,n)_{\alpha]}V(u,n)^{\sigma} = (R_{\Sigma})_{\alpha\beta\rho}{}^{\sigma}V(u,n)^{\rho}, \tag{55}$$

where R_{Σ} is defined by the following first Gauss–Codazzi identity:

$$\begin{aligned}(R_{\Sigma})_{\alpha\beta\rho\sigma} &= R(u,n)_{\alpha\beta\rho\sigma} + K_n(u,n)_{\beta\sigma}K_n(u,n)_{\alpha\rho} - K_n(u,n)_{\alpha\sigma}K_n(u,n)_{\beta\rho}\\ &\quad - \Theta(u,n)_{\beta\sigma}\Theta(u,n)_{\alpha\rho} + \Theta(u,n)_{\alpha\sigma}\Theta(u,n)_{\beta\rho}.\end{aligned} \tag{56}$$

Furthermore, we have the following pair of second Gauss–Codazzi identities:

$$\begin{aligned}&h(u,n)_{\alpha}{}^{\nu}h(u,n)_{\beta}{}^{\mu}h(u,n)_{\sigma}{}^{\gamma}n^{\rho}R_{\nu\mu\rho\gamma}\\ &\quad = 2\nabla(u,n)_{[\beta}K_n(u,n)_{\alpha]\sigma} + 2(\Theta(u,n)_{[\alpha} - \Omega(u,n)_{[\alpha})\Theta(u,n)_{\beta]\sigma},\end{aligned} \tag{57}$$

and

$$\begin{aligned}&h(u,n)_{\alpha}{}^{\nu}h(u,n)_{\beta}{}^{\mu}h(u,n)_{\sigma}{}^{\gamma}u^{\rho}R_{\nu\mu\rho\gamma}\\ &\quad = 2\nabla(u,n)_{[\beta}\Theta(u,n)_{\alpha]\sigma} + 2(\Theta(u,n)_{[\alpha} - \Omega(u,n)_{[\alpha})K_n(u,n)_{\beta]\sigma}.\end{aligned} \tag{58}$$

In the literature null hypersurface Gauss–Codazzi analogues are sometimes introduced by means of auxiliary screen-distributions (see, e.g. [2,8,11]). If Σ is a null hypersurface with normal vector ℓ and u a reference frame, the tangent vector space $T\Sigma$ can be represented in the following way:

$$T\Sigma = \langle L, a(u,n), b(u,n)\rangle, \tag{59}$$

where we recall that $n = n(u)$ and $L = u+n$ obviously depend on the choice of u. However, representation (59) is formally invariant with respect to the choice of u. We thus see that a screen-distribution, i.e. a completion of $T\Sigma$ with respect to ℓ (or L), is given by

$$S\Sigma = \langle a(u,n), b(u,n)\rangle. \tag{60}$$

Clearly, this identifies an observer-dependent, but formally invariant, transverse screen-distribution; thus the choice of the screen-distribution can be reduced to that of the reference frame.

It will be interesting to investigate in detail the corresponding relations between our transverse extrinsic curvature, obtained with the transverse-splitting method, and the different analogous fields obtained in the literature with the screen-distribution method. Moreover, in principle it is possible to apply the transverse-splitting framework to the study of totally geodesic degenerate hypersurfaces, as well of umbilical submanifolds of dimension 2 (see, e.g. [14,15]). However, for reasons of space this study will be eventually included in a forthcoming paper.

5. Concluding remarks

We developed a transverse-splitting formalism and obtained a corresponding Gauss–Codazzi framework for null hypersurfaces. Besides the generality, this method undoubtedly has the good qualities of being rather simple, to give a sort of physical interpretation (in terms of reference frames) to the degree of freedom we always meet while handling with null hypersurfaces, and to completely lie in the framework of tensor algebra and of generic coordinate charts of the ambient space–time.

Acknowledgements

The author gratefully acknowledges the help of Prof. F. de Felice for the critical reading of the manuscript and a number of useful suggestions.

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